Ministerstwo Nauki i Szkolnictwa Wyższego

Organizacja IX Konferencji Modelowanie Matematyczne w Fizyce i Technice (MMFT 2017)
- zadanie finansowane w ramach umowy 829/P-DUN/2017 ze środków Ministra Nauki i Szkolnictwa Wyższego przeznaczonych na działalność upowszechniającą naukę.

POLITECHNIKA CZĘSTOCHOWSKA

TRIDIAGONAL TOEPLITZ SYSTEMS: APPROACH BASED ON LINEAR RECURRENCES VERSUS THOMAS METHOD

Jolanta Borowska, Institute of Mathematics, Czestochowa University of Technology

FORMULATION OF THE PROBLEM

The subject of considerations are linear systems of algebraic equations of a tridiagonal Toeplitz type. The subsequent analysis will be restricted to the systems which have the unique solutions. We are to compare the two methods: approach based on linear recurrences and Thomas algorithm. First of them was proposed for the general tridiagonal system in [1] where the corresponding recurrence equations are shown. Thomas algorithm is well known in literature, [2,3].

A linear algebraic tridiagonal Toeplitz system for n unknowns has the form

$$\begin{cases} ax_1 + cx_2 = d_1 \\ bx_{k-1} + ax_k + cx_{k+1} = d_k, \quad k = 2,\dots,n-1 \\ bx_{n-1} + ax_n = d_n \end{cases} \tag{1}$$

It is convenient to represent system (1) by the corresponding matrix equation

$$\mathbf{A}_n \cdot \mathbf{x} = \mathbf{d} \tag{2}$$

where

$$\mathbf{A}_n = \begin{bmatrix} a & c & 0 & \dots & \dots & 0 \\ b & a & c & \ddots & & \vdots \\ 0 & b & a & c & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & b & a & c \\ 0 & \dots & \dots & 0 & b & a \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}, \mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ d_{n-1} \\ d_n \end{bmatrix}$$

METHOD BASED ON LINEAR RECURRENCES

W_n - determinant of matrix $\mathbf{A}_n$.

$W_n^{x_1}$- determinant of the matrix obtained from matrix $\mathbf{A}_n$ by replacing its first column by corresponding elements of vector ***d***.

In order to obtain solution to system (1) we need to solve three linear recurrence equations together with pertinent initial conditions.

Second order linear homogeneous recurrence equation

$$W_n - aW_{n-1} + bcW_{n-2} = 0, \quad n > 2 \tag{3}$$

with initial conditions

$$\begin{cases} W_1 = a, \\ W_2 = a^2 - bc \end{cases} \tag{4}$$

Second order linear nonhomogeneous recurrence equation

$$W_n^{x_1} - aW_{n-1}^{x_1} + bcW_{n-2}^{x_1} = (-c)^{n-1} d_n, \quad n > 2 \tag{5}$$

with initial conditions

$$\begin{cases} W_1^{x_1} = d_1, \\ W_2^{x_1} = ad_1 - cd_2 \end{cases} \tag{6}$$

Second order linear nonhomogeneous recurrence equation

$$cx_k + ax_{k-1} + bx_{k-2} = d_{k-1} \tag{7}$$

with initial conditions

$$\begin{cases} x_1 = \dfrac{W_n^{x_1}}{W_n}, \\ x_2 = \dfrac{1}{c}(d_1 - ax_1) \end{cases} \tag{8}$$

THOMAS METHOD

Bearing in mind [2] we conclude that solution to system of linear equations (1) can be obtained in two steps.

Firstly, we calculate coefficients α_k, β_k from the system of recurrence equations

$$\begin{cases} \alpha_i = -\dfrac{c}{b\alpha_{i-1} + a} \\ \beta_i = \dfrac{d_i - b\beta_{i-1}}{b\alpha_{i-1} + a} \end{cases}$$

with initial conditions

$$\begin{cases} \alpha_1 = -\dfrac{c}{a}, \\ \beta_1 = \dfrac{d_1}{a} \end{cases}$$

Secondly, we calculate unknowns x_k of the system (1). It can be proved that x_k, k=1,2,…,n satisfies the recurrence relations of the form

$$\begin{cases} x_n = \beta_n \\ x_k = \alpha_k x_{k+1} + \beta_k, \quad k = n-1, n-2, \dots, 1 \end{cases}$$

It can be underline that Thomas algorithm is not stable in general. It can be successfully used when the matrix $\boldsymbol{A_n}$ is diagonally dominant or symmetric positive definite, [2]. The characterization of stability of this algorithm can be found in [3].

EXAMPLE

Now, let as illustrate the two above presented approaches by a certain special case. To this end let us consider the following system of linear equations

$$\begin{cases} 3x_1 + 2x_2 = 1 \\ x_{k-1} + 3x_k + 2x_{k+1} = k, \quad k = 2,\dots,n-1 \\ x_{n-1} + 3x_n = n \end{cases} \tag{9}$$

SOLUTION (RECURRENCE APPROACH)

In this case formulae (3)-(8) take the form

$$W_n = 3W_{n-1} - 2W_{n-2}$$

$$\begin{cases} W_1 = 3 \\ W_2 = 7 \end{cases}$$

$$W_n^{x_1} = 3W_{n-1}^{x_1} - 2W_{n-2}^{x_1} + n \cdot (-2)^{n-1}$$

$$\begin{cases} W_1^{x_1} = 1 \\ W_2^{x_1} = -1 \end{cases}$$

$$x_k = \frac{1}{2}(k - 1 - 3x_{k-1} - x_{k-2})$$

$$\begin{cases} x_1 = -\dfrac{1}{9 \cdot (2^{n+1} - 1)}\left(1 - 9 \cdot 2^{n-2} + (6n+5) \cdot (-2)^{n-2}\right) \\ x_2 = \dfrac{1}{2}(1 - 3x_1) \end{cases}$$

At the end we come to the closed form of unknowns x_k, k=1,2,…,n of system (9)

$$x_k = \frac{1}{36(2^{n+1}-1)}\left((-1)^{k+n} 2^{n+1}(6n+5)\left(1 - \left(\frac{1}{2}\right)^k\right) + (6k-1)(2^{n+1}-1) + (-1)^k (2^{n-k+1}-1)\right)$$

SOLUTION (THOMAS METHOD)

Now, let us apply the Thomas algorithm. It can be seen that this approach doesn't enable to obtain the closed form of solution. We are to implement the Thomas algorithm to Maple**.** Let us assume that the number of unknowns in system (12) is equal 1000.

Firstly we declare all data

$n := 1000$:

$a := Array([seq(3, i = 1..n)])$:

$b := Array([0, seq(1, i = 2..n)])$:

$c := Array([seq(2, i = 1..n-1), 0])$:

$d := Array([seq(j, i = 1..n)])$:

$\alpha := Array([seq(0, i = 1..n)])$:

$\beta := Array([seq(0, i = 1..n)])$:

Afterwards we calculate coefficients α_k, β_k

$\alpha[1] := -\dfrac{c[1]}{a[1]}$: $\beta[1] := \dfrac{d[1]}{a[1]}$:

for i **from** 2 **to** n **do**

$\alpha[i] := -\dfrac{c[i]}{b[i] \cdot \alpha[i-1] + a[i]}$:

$\beta[i] := \dfrac{d[i] - b[i] \cdot \beta[i-1]}{b[i] \cdot \alpha[i-1] + a[i]}$:

end do:

Finally, we calculate unknowns x_k, k=1,2,…,1000 of the system (9)

$x := Array([seq(0, i = 1..n)])$:

$x[n] := \beta[n]$:

for j **from** 1 **to** $n-1$ **do**

$x[n-j] := \alpha[n-j] \cdot x[n+1-j] + \beta[n-j]$

end do:

$print(x)$

CONCLUSIONS

It can be pointed out that we have obtained the same values of x_k, k=1,…,1000 when we used the above two methods. The advantage of the first of the proposed methods is that it enables us to obtain solution in the compact form.

REFERENCES

1) Borowska J., Łacińska L., Application of second order inhomogeneous linear recurrences to solving a tridiagonal system, Journal of Applied Mathematics and Computational Mechanics, 2016, 15(2), 5-10.
2) Datta B., N., Numerical Linear Algebra and Applications: Second Edition, SIAM 2010.
3) Higham N., J., Accuracy and Stability of Numerical Algorithms: Second Edition, SIAM, 2002.